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Third-order Kalman Filter : tuning and steady-state performance

Huaqiang Shu, Eric Pierre Simon, and Laurent Ros

***Abstract*—This letter deals with the Kalman filter (KF) based on a third-order integrated random walk model (RW3). The resulting filter, noted as RW3-KF, is well suited to track slow time-varying parameters with strong trend behaviour. We first prove that the RW3-KF in steady-state admits an equivalent structure to the third-order digital phase-locked loops (DPLL). The approximate asymptotic mean-squared-error (MSE) is obtained by solving the Riccati equations, which is given in a closed-form expression as a function of the RW3 model parameter: the state noise variance. Then, the closed-form expression of the optimum state noise variance is derived to minimize the asymptotic MSE. Simulation results are given for the particular case where the parameter to be estimated is a Rayleigh channel coefficient with Jakes' Doppler spectrum.**

***Index Terms*—Random Walk model (RW), Kalman filter (KF).**

I. INTRODUCTION

Kalman filters (KF) are commonly used to track time-varying parameters. The applications of KF cover a various range of systems, like GPS systems [1], Multi-carrier systems [2], MIMO systems [3], etc. The design of KF requires a linear recursive state-space representation of the parameter to be observed. The most used approximation model, especially for channel estimation problems, is the first-order Auto-Regressive model (AR1), combined with either a correlation matching (CM) criterion for the fast time-varying scenario [2] [4], or a minimum asymptotic variance (MAV) criterion for the slow varying scenario [5], to fix the AR coefficient. However, in certain systems, the parameter to be estimated exhibits strong trend behaviour, and the use of second-order or higher-order models is more suitable than a first-order model. For example in a satellite receiver, third-order KF as well as third-order DPLLs are often used to tackle the problem of phase tracking in the presence of time-varying Doppler frequency offset [1]. However, the tuning and performance of these estimators are most often obtained from simulation or empirical results.

In this paper, we provide analytic results about the optimal tuning and the steady-state performance of a KF based on a RW3 model. For that, we first prove that this third-order KF has the same structure in a steady-state mode as a specific equal-order DPLL, hence extending the results of [6] [7] obtained for a second-order KF.

Section II gives the approximation model and the formulae of RW3-KF. In section III, we analyze and optimize the asymptotic MSE of RW3-KF. Section IV validates the analysis and assumptions by means of MSE and BER (bit error rate) simulations, the first-order AR model-based KFs (combined with CM and MAV criterion, respectively noted as AR1$_{CM}$-KF and AR1$_{MAV}$-KF) are selected as references.

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II. STATE-SPACE MODEL AND KALMAN FILTER

Assume the parameter to be estimated α is a zero-mean circular complex process with variance σ_α^2. The variable α is supposed to be a narrow-band stationary process, with a Power Spectrum Density (PSD) $\Gamma_\alpha(f)$ with a support limited within $\pm f_d$. We consider the RW3 model as an approximation of the time-variation of α:

$$\tilde{\alpha}_{(n)} = \tilde{\alpha}_{(n-1)} + \delta_{(n-1)} + \frac{1}{2}\xi_{(n-1)}, \tag{1}$$

$$\delta_{(n)} = \delta_{(n-1)} + \xi_{(n-1)}, \tag{2}$$

$$\xi_{(n)} = \xi_{(n-1)} + u_{(n)}, \tag{3}$$

where $u_{(n)}$ is the state noise, a zero mean complex state noise with variance σ_u^2. The model is updated at sample rate. The time interval between each sample, T, represents a unit delay. A simplistic observation model is used [1]:

$$y_{(n)} = \alpha_{(n)} + \mathrm{w}_{(n)}, \tag{4}$$

where $\mathrm{w}_{(n)}$ is a zero-mean additive white noise with variance σ_{w}^2. The dynamic evolution equations (1)-(3) and the observation equation (4) compose the state-space model of $\alpha_{(n)}$. The on-line unbiased estimation $\hat{\alpha}_{(n)}$ can be carried out by KF. The MSE $\sigma_\epsilon^2 \overset{\text{def}}{=} E\left\{\left|\epsilon_{(n)}\right|^2\right\}$ of the estimation error $\epsilon_{(n)} \overset{\text{def}}{=} \alpha_{(n)} - \hat{\alpha}_{(n)}$ will be investigated.

Rewrite the state-space model in the matrix form:

$$\boldsymbol{a}_{(n)} = \mathbf{M}\boldsymbol{a}_{(n-1)} + \mathbf{u}_{(n)}, \tag{5}$$

$$y_{(n)} = \mathbf{S}\boldsymbol{a}_{(n)} + \mathrm{w}_{(n)}, \tag{6}$$

with the state vector $\boldsymbol{a}_{(n)} = \begin{bmatrix}\tilde{\alpha}_{(n)} & \delta_{(n)} & \xi_{(n)}\end{bmatrix}^T$, the state noise vector $\mathbf{u}_{(n)} = \begin{bmatrix}0 & 0 & u_{(n)}\end{bmatrix}^T$, the selection vector $\mathbf{S} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}$ and the evolution matrix $\mathbf{M} = \begin{bmatrix}1 & 1 & \frac{1}{2}\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}$, the RW3-KF could then described by two-stage equations:

Time Update Equations

$$\hat{\boldsymbol{a}}_{(n|n-1)} = \mathbf{M}\hat{\boldsymbol{a}}_{(n-1|n-1)}, \tag{7}$$

$$\mathbf{P}_{(n|n-1)} = \mathbf{M}\mathbf{P}_{(n-1|n-1)}\mathbf{M}^T + \mathbf{U}, \tag{8}$$

[1]This model is adequate for many applications, *e.g.* it could be a flat fading channel model, α is then the complex amplitude of channel; or in the vehicle tracking problem, α could be in matrix form, composed by the position coordinates and velocities of vehicle. etc.

Measurement Update Equations

$$\mathbf{K}_{(n)} = \frac{\mathbf{P}_{(n|n-1)}\mathbf{S}^T}{\mathbf{S}\mathbf{P}_{(n|n-1)}\mathbf{S}^T + \sigma_{\mathrm{w}}^2}, \tag{9}$$

$$\hat{\boldsymbol{a}}_{(n|n)} = \hat{\boldsymbol{a}}_{(n|n-1)} + \mathbf{K}_{(n)}(y_{(n)} - \mathbf{S}\hat{\boldsymbol{a}}_{(n|n-1)}), \tag{10}$$

$$\mathbf{P}_{(n|n)} = (\mathbf{I} - \mathbf{K}_{(n)}\mathbf{S})\mathbf{P}_{(n|n-1)}, \tag{11}$$

with the Kalman gain $\mathbf{K}_{(n)} = \begin{bmatrix} k_{1(n)} & k_{2(n)} & k_{3(n)} \end{bmatrix}^T$, the state noise variance matrix $\mathbf{U} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_u^2 \end{bmatrix}$, $\mathbf{P}_{(n|n-1)}$ and $\mathbf{P}_{(n|n)}$ are respectively the covariance matrices of the prediction error and the estimation error.

III. Asymptotic MSE analysis

A. Steady-state RW3-KF

Since the linear model ((5),(6)) is *observable* and *controllable*, an asymptotic regime is quickly reached ([8] Ch. 13.3). In other words, $\mathbf{P}_{(n|n)}$,$\mathbf{P}_{(n|n-1)}$ and $\mathbf{K}_{(n)}$ converge to constant values when n is large enough, *i.e.*,

$$\mathbf{K}_{(n)} = \mathbf{K}_{(n+1)} = \mathbf{K}_{(\infty)} \overset{\text{def}}{=} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}^T, \tag{12}$$

$$\mathbf{P}_{(n|n)} = \mathbf{P}_{(n+1|n+1)} = \mathbf{P}_{(\infty)} \overset{\text{def}}{=} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}, \tag{13}$$

$$\mathbf{P}_{(n|n-1)} = \mathbf{P}_{(n+1|n)} = \mathbf{P}'_{(\infty)} \overset{\text{def}}{=} \begin{bmatrix} P'_{11} & P'_{12} & P'_{13} \\ P'_{21} & P'_{22} & P'_{23} \\ P'_{31} & P'_{32} & P'_{33} \end{bmatrix}. \tag{14}$$

Note that $\mathbf{P}_{(\infty)}$ and $\mathbf{P}'_{(\infty)}$ are real symmetric matrices. This can be easily verified from (8), (9), (11) if the KF starts with a real-valued matrix $\mathbf{P}_{(0|-1)}$. $\mathbf{K}_{(\infty)}$ is also a real vector. In the steady state, from (7) and (10), the recursive equations of RW3-KF can be reduced to a time-invariant filter:

$$\hat{\alpha}_{(n|n)} = \hat{\alpha}_{(n-1|n-1)} + \hat{\delta}_{(n-1|n-1)} + \frac{1}{2}\hat{\xi}_{(n-1|n-1)} + k_1 v_{\epsilon(n)}, \tag{15}$$

$$\hat{\delta}_{(n|n)} = \hat{\delta}_{(n-1|n-1)} + \hat{\xi}_{(n-1|n-1)} + k_2 v_{\epsilon(n)}, \tag{16}$$

$$\hat{\xi}_{(n|n)} = \hat{\xi}_{(n-1|n-1)} + k_3 v_{\epsilon(n)}, \tag{17}$$

with

$$v_{\epsilon(n)} = y_{(n)} - (\hat{\alpha}_{(n-1|n-1)} + \hat{\delta}_{(n-1|n-1)} + \frac{1}{2}\hat{\xi}_{(n-1|n-1)}). \tag{18}$$

Transforming (15), (16), (17) to Z-domain and substituting $\hat{\delta}$ and $\hat{\xi}$ yield:

$$\hat{\alpha}(z)(1-z^{-1}) = \left[k_1 + \frac{(k_2 + \frac{1}{2}k_3)z^{-1}}{1 - z^{-1}} + \frac{k_3 z^{-2}}{(1-z^{-1})^2}\right] v_\epsilon(z). \tag{19}$$

Combining (18), (15) and (4), and after Z-transform we have:

$$v_\epsilon(z) = \frac{1}{1-k_1} \cdot (\alpha(z) - \hat{\alpha}(z) + \mathrm{w}(z)), \tag{20}$$

then substitute (20) into (19), we obtain the input-output equation:

$$\hat{\alpha}(z) = L(z) \cdot \alpha(z) + L(z) \cdot \mathrm{w}(z), \tag{21}$$

with $L(z)$ the transfer function of steady-state RW3-KF given in (22).

In the slow fading scenario ($f_d T \ll 1$), we are interested in the low frequency domain part of $L(z)$ ($fT \ll 1$), using the approximation $1 - z^{-1} \approx pT$, with $z = e^{pT}$ and $p = j2\pi f$. With such an approximation, the steady state transfer function of the RW3-KF $L(e^{j2\pi f})$ is equivalent to the typical transfer function of the third-order analog PLL ([9], eqn. (2), (4) and [10] eqn. (22)):

$$L(e^{pT}) \approx \frac{(m+2)\zeta\omega_n \cdot p^2 + (1+2m\zeta^2)\omega_n^2 \cdot p + m\zeta\omega_n^3}{p^3 + (m+2)\zeta\omega_n \cdot p^2 + (1+2m\zeta^2)\omega_n^2 \cdot p + m\zeta\omega_n^3} \tag{23}$$

with

$$k_1 = \frac{(m+2)\zeta\omega_n T + (1+2m\zeta^2)(\omega_n T)^2 + m\zeta(\omega_n T)^3}{1 + (m+2)\zeta\omega_n T + (1+2m\zeta^2)(\omega_n T)^2 + m\zeta(\omega_n T)^3}, \tag{24}$$

$$k_2 = \frac{(1+2m\zeta^2)(\omega_n T)^2 + \frac{3}{2}m\zeta(\omega_n T)^3}{1 + (m+2)\zeta\omega_n T + (1+2m\zeta^2)(\omega_n T)^2 + m\zeta(\omega_n T)^3}, \tag{25}$$

$$k_3 = \frac{m\zeta(\omega_n T)^3}{1 + (m+2)\zeta\omega_n T + (1+2m\zeta^2)(\omega_n T)^2 + m\zeta(\omega_n T)^3}, \tag{26}$$

and with m the capacitance ratio, ζ the damping factor, $\omega_n = 2\pi f_n$ the natural radian frequency of the loop. They are real positive physical parameters, and $\omega_n T \ll 1$ when assuming a slow reaction of the filter. A useful inequality could be obtained with

$$0 < k_3 \ll k_2 \ll k_1 < 1. \tag{27}$$

This inequality is obtained by comparing the numerators of (24), (25), (26), using $\omega_n T \ll 1$.

We aim to find the relation between the Kalman gains k_1, k_2, k_3, and the state noise variance σ_u^2, in order to optimize the estimation error σ_ϵ^2 with respect to σ_u^2. From (9),

$$k_1 = \frac{P'_{11}}{P'_{11} + \sigma_{\mathrm{w}}^2}, \quad k_2 = \frac{P'_{21}}{P'_{11} + \sigma_{\mathrm{w}}^2}, \quad k_3 = \frac{P'_{31}}{P'_{11} + \sigma_{\mathrm{w}}^2}. \tag{28}$$

From (11), (8) and by using the symmetry of $\mathbf{P}_{(\infty)}$ and $\mathbf{P}'_{(\infty)}$, we have:

$$\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12} & P_{22} & P_{23} \\ P_{13} & P_{23} & P_{33} \end{bmatrix} = \begin{bmatrix} (1-k_1)P'_{11} & (1-k_1)P'_{12} & (1-k_1)P'_{13} \\ P'_{12} - k_2 P'_{11} & P'_{22} - k_2 P'_{12} & P'_{23} - k_2 P'_{13} \\ P'_{13} - k_3 P'_{11} & P'_{23} - k_3 P'_{12} & P'_{33} - k_3 P'_{13} \end{bmatrix}, \tag{29}$$

with $P'_{11} = P_{11} + 2P_{12} + P_{13} + P_{22} + P_{23} + \frac{1}{4}P_{33}$; $P'_{12} = P_{12} + P_{22} + \frac{3}{2}P_{23} + P_{13} + \frac{1}{2}P_{33}$; $P'_{22} = P_{22} + 2P_{23} + P_{33}$; $P'_{13} = P_{13} + P_{23} + \frac{1}{2}P_{33}$; $P'_{23} = P_{23} + P_{33}$; $P'_{33} = P_{33} + \sigma_u^2$. The equations (28), (29) compose the so called Riccati equations.

$$L(z) = \frac{(k_1 - k_2 + \frac{k_3}{2})(1-z^{-1})^2 + (k_2 - \frac{k_3}{2})(1-z^{-1}) + k_3}{(1-k_1)(1-z^{-1})^3 + (k_1 - k_2 + \frac{k_3}{2})(1-z^{-1})^2 + (k_2 - \frac{k_3}{2})(1-z^{-1}) + k_3}. \tag{22}$$

By solving these equations, we could find the expressions of the elements of $\mathbf{P}^{'}_{(\infty)}$ as a function of k_1, k_2, k_3, and σ_u^2. To this end, after some manipulations on (29), we first find:

$$P^{'}_{13} = P^{'}_{31} = \frac{\sigma_u^2}{k_3}, \tag{30}$$

which enables us to find after some manipulations:

$$P^{'}_{11} = \frac{8k_2 + k_3(6k_1 + 3k_2 + k_3)}{2k_3^2(2k_1 + 2k_2 + k_3)}\sigma_u^2. \tag{31}$$

Then, a relation between k_1, k_2 and k_3 is found:

$$k_2^2 = 2k_1k_3. \tag{32}$$

In the sequel, it is assumed that $P^{'}_{11} \ll \sigma_{\mathrm{w}}^2$, which means that the Kalman gain is low $k_1 \ll 1$, according to (28). Then we deduce k_1 from (28), k_3 from (28) and (30) and k_2 from (32) respectively, that is:

$$k_1 \approx \frac{P^{'}_{11}}{\sigma_{\mathrm{w}}^2},\ k_3 = \frac{\sigma_u}{\sqrt{P^{'}_{11} + \sigma_{\mathrm{w}}^2}} \approx \frac{\sigma_u}{\sigma_{\mathrm{w}}},\ k_2 \approx \sqrt{\frac{2P^{'}_{11}\sigma_u}{\sigma_{\mathrm{w}}^3}}. \tag{33}$$

To further simplify the calculation, we apply the approximation (27) on (31), yielding:

$$P^{'}_{11} \approx \frac{2k_2}{k_1k_3^2}\sigma_u^2. \tag{34}$$

By combining (33), (34), $P^{'}_{11}$ can be expressed as a function of σ_u and σ_{w}:

$$P^{'}_{11} \approx 2\sigma_u^{\frac{1}{3}}\sigma_{\mathrm{w}}^{\frac{5}{3}} = 2\sigma_{\mathrm{w}}^2(\frac{\sigma_u}{\sigma_{\mathrm{w}}})^{\frac{1}{3}}, \tag{35}$$

and finally,

$$k_1 \approx 2(\frac{\sigma_u}{\sigma_{\mathrm{w}}})^{\frac{1}{3}},\ k_2 \approx 2(\frac{\sigma_u}{\sigma_{\mathrm{w}}})^{\frac{2}{3}} = \frac{k_1^2}{2},\ k_3 \approx \frac{\sigma_u}{\sigma_{\mathrm{w}}} = \frac{k_1^3}{8}. \tag{36}$$

Using the approximated Kalman gain relation $0 < k_3 \ll k_2 \ll k_1 \ll 1$, the transfer function of RW3-KF (22) can be simplified as:

$$L(z) \approx \frac{k_1(1-z^{-1})^2 + k_2(1-z^{-1}) + k_3}{(1-z^{-1})^3 + k_1(1-z^{-1})^2 + k_2(1-z^{-1}) + k_3}. \tag{37}$$

Comparing (37) and (23), we get $k_1 \approx (m+2)\zeta\omega_nT$, $k_2 \approx (1+2m\zeta^2)(\omega_nT)^2$, $k_3 \approx m\zeta(\omega_nT)^3$. Then by using (36), we obtain $m = 2$, $\zeta = 0.5$, while its natural radian frequency ω_nT can be tuned as $\frac{k_1}{2}$, or eventually $(\frac{\sigma_u}{\sigma_{\mathrm{w}}})^{\frac{1}{3}}$. Thus, we can conclude that the RW3-KF is equivalent in steady-state mode and slow-tracking scenario to the third-order DPLL with fixed given parameters ($m = 2$, $\zeta = 0.5$). This conclusion generalizes to the third-order the connection between DPLL and KF established in [6] [7] for the second-order.

B. Mean Squared Error Analysis

The (unbiased) estimation error is defined by:

$$\epsilon(z) = \alpha(z) - \hat{\alpha}(z) = (1 - L(z))\cdot\alpha(z) - L(z)\cdot \mathrm{w}(z) \tag{38}$$

and the mean squared error is thus composed by two parts:

$$\sigma_\epsilon^2 = E\{\epsilon\cdot\epsilon^*\} = \sigma_{\epsilon\alpha}^2 + \sigma_{\epsilon\mathrm{w}}^2, \tag{39}$$

$\sigma_{\epsilon\mathrm{w}}^2$ is the static error variance which results from the channel noise w, whereas $\sigma_{\epsilon\alpha}^2$ is the dynamic error variance, which results from the parameter α variations.

The static error variance is developed as:

$$\begin{aligned}\sigma_{\epsilon\mathrm{w}}^2 &= \sigma_{\mathrm{w}}^2 \cdot \underbrace{T\int_{-\frac{1}{2T}}^{+\frac{1}{2T}} |L(e^{j2\pi fT})|^2 df}_{B_L} \\ &= \frac{\frac{5}{2}k_1 - \frac{1}{4}k_1^2 - \frac{5}{8}k_1^3 - \frac{5}{64}k_1^4}{3 - \frac{9}{8}k_1^2 - \frac{9}{32}k_1^3 - \frac{3}{64}k_1^4}\sigma_{\mathrm{w}}^2 \approx \frac{5}{3}k_3^{\frac{1}{3}}\sigma_{\mathrm{w}}^2,\end{aligned} \tag{40}$$

where the integral term B_L is the equivalent noise bandwidth. It can be calculated by the method presented in [11]. Note that we have applied the condition $0 < k_1 \ll 1$ for the approximation. The dynamic error variance is developed as:

$$\begin{aligned}\sigma_{\epsilon\alpha}^2 &= \int_{-\frac{1}{2T}}^{+\frac{1}{2T}} \Gamma_\alpha(f)\cdot|1 - L(e^{j2\pi fT})|^2 df \\ &\approx \int_{-\frac{1}{2T}}^{+\frac{1}{2T}} \Gamma_\alpha(f)\cdot\frac{(2\pi fT)^6}{k_3^2}df = \frac{(2\pi)^6}{k_3^2}S_\alpha,\end{aligned} \tag{41}$$

where $S_\alpha = \int_{-\frac{1}{2T}}^{+\frac{1}{2T}} \Gamma_\alpha(f)(fT)^6 df$ is the term which contains the PSD of α. For the reason of simplicity, we apply $e^{-j2\pi fT} \approx 1 - j2\pi fT$ as well as $2\pi f_dT \ll k_3^{\frac{1}{3}} \approx \omega_nT \ll 1$ to calculate $|1 - L(e^{j2\pi fT})|^2$ in the slow variation channel case. The global MSE is then obtained by combining (40) and (41). After substituting the approximation (36) for k_3, the objective function to optimize is given by:

$$\sigma_\epsilon^2 = \frac{5}{3}\sigma_{\mathrm{w}}^{\frac{5}{3}}\sigma_u^{\frac{1}{3}} + (2\pi)^6 S_\alpha\frac{\sigma_{\mathrm{w}}^2}{\sigma_u^2}. \tag{42}$$

The minimization can be done by imposing the partial derivative of global MSE σ_ϵ^2 equal to 0, yielding:

$$\sigma_{u\ \mathrm{opt}}^2 = \left[(2\pi)^{36}\cdot(\frac{18}{5}S_\alpha)^6\cdot\sigma_{\mathrm{w}}^2\right]^{\frac{1}{7}}, \tag{43}$$

and the corresponding minimized MSE is:

$$\sigma_{\epsilon\ \min}^2 = 7\cdot\left(\frac{5}{9}\pi\cdot\sigma_{\mathrm{w}}^2\right)^{\frac{6}{7}}\cdot S_\alpha^{\frac{1}{7}}. \tag{44}$$

C. An application to the estimation of Rayleigh channel with Jakes' Doppler spectrum

From (43) and (44), we note that the optimum parameter and the corresponding minimized MSE could be computed whatever the channel PSD is. Now we take the estimation of Rayleigh channel with Jakes' Doppler spectrum as an example. The PSD of α is defined as:

$$\Gamma_\alpha(f) = \begin{cases} \dfrac{\sigma_\alpha^2}{\pi f_d\sqrt{1-\left(\frac{f}{f_d}\right)^2}}, & \text{for} \quad |f| < f_d, \\ 0, & \text{for} \quad |f| \geq f_d. \end{cases} \tag{45}$$

A variable change $\cos\theta = \frac{f}{f_d}$ is applied to calculate the integral S_α and we have:

$$S_\alpha = \int_{-f_d}^{+f_d} \frac{(fT)^6\cdot\sigma_\alpha^2}{\pi f_d\sqrt{1-\left(\frac{f}{f_d}\right)^2}}df = \frac{5}{16}(f_dT)^6\cdot\sigma_\alpha^2. \tag{46}$$

The optimal σ_u^2 and the corresponding minimized MSE are then obtained directly:

$$\sigma_{u\ \mathrm{opt}}^2 = \left[\frac{3^{12}}{2^{18}}\cdot(\sigma_\alpha^2)^6\cdot\sigma_{\mathrm{w}}^2\cdot(2\pi f_dT)^{36}\right]^{\frac{1}{7}}, \tag{47}$$

$$\sigma_{\epsilon\ \min}^2 = \frac{35}{16}\cdot(\frac{16}{9}\pi f_dT\cdot\sigma_{\mathrm{w}}^2)^{\frac{6}{7}}\cdot(\sigma_\alpha^2)^{\frac{1}{7}}. \tag{48}$$

IV. Simulation results and conclusion

The MSE analysis is verified by Monte-Carlo simulations over a Rayleigh flat fading channel. Fig. 1 shows the MSE of AR1$_{CM}$-KF [2] [3] [4], AR1$_{MAV}$-KF [5] and RW3-KF as a function of SNR, with $f_dT = 10^{-3}$. The theoretical MSE of RW3-KF as well as the online BCRB (Bayesian Cramer-Rao Bound) [12] are used as references. Fig. 2 shows the MSE of these estimators as a function of f_dT with fixed SNR=20dB. From Fig. 1 and 2, we find that the theoretical and the simulation lines of RW3-KF approximately coincide. The MSE of RW3-KF is proportional to the $\frac{6}{7}$ power of noise variance σ_{w}^2 (thus inversely proportional to the SNR), and is also proportional to the $\frac{6}{7}$ power of f_dT. On the other hand, compared to the AR1$_{CM}$-KF, the AR1$_{MAV}$-KF has a much improved asymptotic performance, which means that the MAV criterion is a better choice for computing the AR1 coefficient. However it is still far from the lower bound due to the low-order filtering that causes the loss of dynamic information. Meanwhile, the RW3 model fits the real channel much better than the AR models in the slow fading case. Moreover, the MSE of RW3-KF is very close to the online BCRB.

For the BER simulation, we use QPSK transmitted symbols. The estimation is in semi-blind mode, that is, the data block is composed of 20 pilot symbols followed by 180 unknown symbols (for which the KF is in decision-directed mode). Fig.3 shows the simulation result, where we can observe that with the optimized $\sigma_{u\ \mathrm{opt}}^2$, the RW3-KF attains a performance close to the one with perfect channel knowledge.

To conclude, we have discussed in this letter the third-order modeling of the Kalman Filter for parameter estimation problems, where an application to Rayleigh fading channel with Jakes' spectrum was also introduced. The explicit formulae of the optimum parameter and the asymptotic MSE of the RW3-KF were given, assuming the knowledge of the channel statistics. A connection between the steady-state RW3-KF and the typical third-order DPLL was established. We also conclude that, for KF-based estimators, the well-tuned third-order random walk model is more adequate compared with the first-order AR model in the low-variation context, with the resulting estimator performance very close to the BCRB. Possible future directions are to extend this work to the vectorial case for multi-path channel and/or multi-carrier modulation scenarios. Also, MSE performance of the other components of the RW3 model could be investigated.

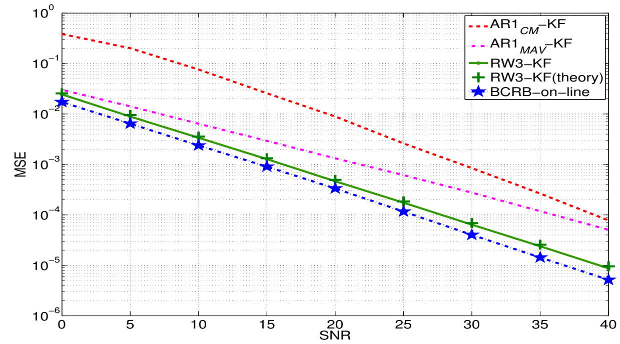

Fig. 1: MSE versus SNR with $f_dT = 10^{-3}$

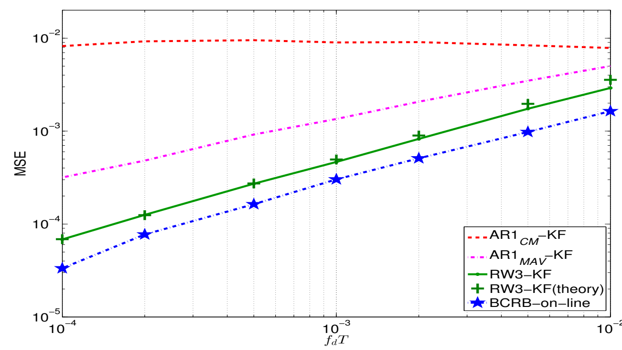

Fig. 2: MSE versus f_dT with SNR = 20 dB

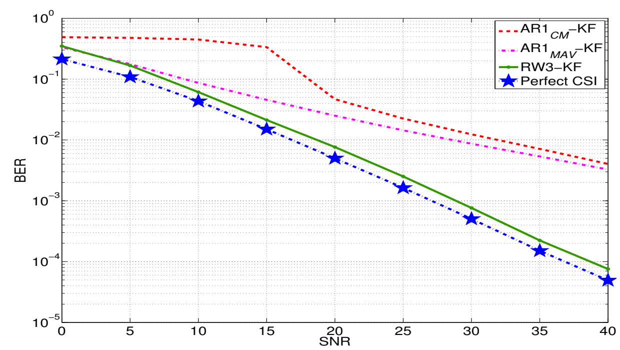

Fig. 3: BER versus SNR for QPSK modulation, $f_dT = 10^{-3}$

References

[1] E. Kaplan and C. Hegarty, *Understanding GPS: Principles and Applications, Second Edition*. Artech House Publishers, 2005.

[2] H. Hijazi and L. Ros, "Joint data QR-detection and Kalman estimation for OFDM time-varying Rayleigh channel complex gains," *IEEE Trans. Commun.*, vol. 58, no. 1, pp. 170 – 178, Jan. 2010.

[3] C. Komninakis, C. Fragouli, A. Sayed, and R. Wesel, "Multi-input multi-output fading channel tracking and equalization using Kalman estimation," *IEEE Trans. Signal Process.*, vol. 50, no. 5, pp. 1065 – 1076, May 2002.

[4] E. P. Simon, L. Ros, H. Hijazi, and M. Ghogho, "Joint Carrier Frequency Offset and Channel Estimation for OFDM Systems via the EM Algorithm in the Presence of Very High Mobility," *IEEE Trans. Signal Process.*, vol. 60, no. 2, pp. 754 – 765, Feb. 2012.

[5] S. Ghandour-Haidar, L. Ros, and J. M. Brossier, "On the use of first-order autoregressive modeling for Rayleigh flat fading channel estimation with Kalman filter," *Signal Processing*, vol. 92, no. 2, pp. 601 – 606, 2012.

[6] P. Driessen, "DPLL bit synchronizer with rapid acquisition using adaptive Kalman filtering techniques," *IEEE Trans. Commun.*, vol. 42, no. 9, pp. 2673 – 2675, sep 1994.

[7] A. Patapoutian, "On phase-locked loops and Kalman filters," *IEEE Trans. Commun.*, vol. 47, no. 5, pp. 670 – 672, May, 1999.

[8] S. M. Kay, *Fundamentals of Statistical Signal Processing : Estimation Theory*. Prentice Hall PTR, April 5, 1993.

[9] C. Shan, Z. Chen, L. Zhu, and Y. Li, "Design and Implementation of Bandwidth Adaptable Third-Order All Digital Phase-Locked Loops," in *6th International Conference on Wireless Communications Networking and Mobile Computing (WiCOM)*, 2010, pp. 1 – 4.

[10] H. Shu, L. Ros, and E. P. Simon, "Third-order complex amplitudes tracking loop for slow fading channel estimation," in *19th International Conference on Telecommunications (ICT)*, April, 2012, pp. 1 – 6.

[11] R.Winkelstein, "Closed form evaluation of symmetric two-sided complex integrals," *TDA Progress Report*, 1981.

[12] H. Hijazi and L. Ros, "Bayesian Cramer-Rao bounds for complex gain parameters estimation of slowly varying Rayleigh channel in OFDM systems," *Signal Processing*, vol. 89, no. 1, pp. 111 – 115, 2009.